

# A Novel Method for Integrating the Multivariate Normal Density

Compiled by Paul Mullenix, Ph.D. from Lectures by the late John G. Saw, Ph.D.

## Introduction

Almost every introductory mathematical statistics theory course requires the student to show that the area beneath the standard normal density function is unity. In a later course where multivariate statistical distributions are introduced, the student may be required to demonstrate a similar property for the multivariate normal distribution.

An article by Bell [1] exposed the limited use of the standard “trick” of squaring the integral of the standard normal density function defined by

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

and then converting to polar coordinates to show that  $I = 1$ . However, the use of polar coordinates can be laborious in practice, especially in the multivariate case.

## A New Paradigm

A more elegant method of showing for this particular case that  $I = 1$ , and a method which readily extends to the multivariate case, involves integrating not over the usual partition

of hyper-rectangles, but instead, over a partition defined by concentric  $m$ -spheres,  $S_m(r)$ , of radius  $r$  centered at the origin, i.e.,  $S_m(r) = \{\underline{x}: \underline{x} \in \mathbb{R}^m, \underline{x}'\underline{x} = r^2\}$ . The only results necessary for this approach are knowledge of the gamma function and the well known volume,  $V_m(r)$  ([2], p. 186), of  $S_m(r)$  given by

$$V_m(r) = \frac{(\pi r^2)^{\frac{m}{2}}}{\Gamma(\frac{m}{2} + 1)} = r^m V(1)$$

The integral of the multivariate normal density function re-expressed as integrating  $r = \sqrt{\underline{x}'\underline{x}}$  over a partition defined by  $S_m(r)$  is

$$\int_{\mathbb{R}^m} \dots \int (2\pi)^{-\frac{m}{2}} e^{-\frac{1}{2}\underline{x}'\underline{x}} d\underline{x} = \int_0^\infty (2\pi)^{-\frac{m}{2}} r^{m-1} e^{-\frac{1}{2}r^2} dV_m(r) \quad (1)$$

Since  $dV_m(r) = m r^{m-1} V_m(1) dr$ , we may write the right hand side of (1) as

$$m V_m(1) \int_0^\infty (2\pi)^{-\frac{m}{2}} r^{m-1} e^{-\frac{1}{2}r^2} dr \quad (2)$$

Now let  $u = r^2/2$  so that (2) becomes

$$m V_m(1) \frac{\pi^{-\frac{m}{2}}}{2} \int_0^\infty u^{\frac{m}{2}-1} e^{-u} du = m \frac{\pi^{\frac{m}{2}}}{\frac{m}{2} \Gamma(\frac{m}{2})} \frac{\pi^{-\frac{m}{2}}}{2} \Gamma(\frac{m}{2}) = 1 \quad (3)$$

Alternately, if one begins with the fact that the multivariate normal is a density function, then since  $V_m(r) = r^m V_m(1)$ , the above argument can be used to provide a quick derivation that the volume of an  $m$ -sphere equals  $(\pi r^2)^{m/2} / \Gamma(1+m/2)$ . The typical derivation of  $V_m(r)$ , (see [2] for example), proceeds tediously by a proof by induction and iterated integrals that

$$V_m(1) = 2 V_{m-1}(1) \int_0^1 (1-t^2)^{\frac{m-1}{2}} dt$$

The surface area of the  $m$ -sphere follows directly by differentiating  $V_m(r)$  with respect to  $r$ .

**Acknowledgment.** The method presented here was part a course lecture by Dr John G. Saw. Dr. Saw challenged his students to think in new ways and this is just one of many such examples. This method is presented here after the death of Dr. Saw (1934-1990) so that those who never had the very great honor of knowing Dr. Saw or the privilege of attending his lectures might benefit from the elegant way he approached even the simplest problem.

## REFERENCES

1. Bell, Denis, "On the Limitations of a Well-Known Integration Technique", *Mathematics Magazine*, Vol. 66, No. 4, October 1993, pp. 243-244.
2. Conway, John B., *Functions of One Complex Variable, Second Edition*, Springer-Verlag: New York, 1978, p. 186.